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Algorithms & Data Structures — Exercise sheet 1 — HS 23

The solutions for this sheet are submitted at the beginning of the exercise class on 2 October 2023.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

Exercise 1.1 *Guess the formula* **(1 point)**.

Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 3a_n - 2$ for $n > 1$. Find a simple closed formula for a_n and prove that a_n follows it using mathematical induction.

Hint: *Write out the first few terms. How fast does the sequence grow?*

Solution:

Writing out the first few terms, we get: $2, 4, 10, 28, 82$, etc. From this sequence, we guess the closed formula

$$a_n = 3^{n-1} + 1.$$

We prove this by induction.

- **Base Case.**
  For $n = 1$ we have
  $$a_1 = 2 = 3^{1-1} + 1,$$
  so it is true for $n = 1$.
- **Induction Hypothesis.**
  We now assume that it is true for $n = k$, i.e., $a_k = 3^{k-1} + 1$.
- **Induction Step.**
  We want to prove that it is also true for $n = k + 1$. Using the induction hypothesis we get
  $$a_{k+1} = 3a_k - 2 = 3 \cdot (3^{k-1} + 1) - 2 = 3 \cdot 3^{k-1} + 3 - 2 = 3^k + 1.$$
  Hence, it is true for $n = k + 1$.

By the principle of mathematical induction, we conclude that $a_n = 3^{n-1} + 1$ is true for any positive integer n.

Exercise 1.2 *Sum of Cubes* **(1 point)**.

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Solution:

- **Base Case.**
  Let $n = 1$. Then,
  $$1^3 = 1 = \frac{1^2 \cdot (1+1)^2}{4},$$
  so the property holds for $n = 1$.
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer k, that is,
  $$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$
- **Induction Step.**
  We must show that the property also holds for $k+1$. Let us add $(k+1)^3$ to both sides of the induction hypothesis. We get
  $$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2 + ((k+1)+1)^2}{4}. \end{aligned}$$
  By the principle of mathematical induction, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ is true for any positive integer n.

Exercise 1.3 *Sums of powers of integers.*

In this exercise, we fix an integer $k \in \mathbb{N}_0$.

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \leq n^{k+1}$.

Solution:

As all terms in the sum are at most n^k, we have:

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}.$$

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$.

Hint: *Consider the second half of the sum, i.e.,* $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$. *How many terms are there in this sum? How small can they be?*

Solution:

We have:

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^k$$

By definition of $\lceil \cdot \rceil$, we have $\lceil \frac{n}{2} \rceil - 1 \leq \frac{n}{2}$, hence $n - \lceil \frac{n}{2} \rceil + 1 \geq \frac{n}{2}$. Hence

$$\sum_{i=1}^{n} i^k \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} \cdot n^{k+1}.$$

Together, these two inequalities show that $C_1 \cdot n^{k+1} \leq \sum_{i=1}^{n} i^k \leq C_2 \cdot n^{k+1}$, where $C_1 = \frac{1}{2^{k+1}}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^k$ behaves "almost like n^{k+1}" up to a constant factor.

Exercise 1.4 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements.

(a) $f(m) = 10m^3 - m^2$ grows asymptotically slower than $g(m) = 100m^3$.

Solution:

False, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{10m^3 - m^2}{100m^3} \\ &= \lim_{m\to\infty} \frac{1}{10} - \frac{1}{100m} = \frac{1}{10} + 0 > 0.\end{aligned}$$

(b) $f(m) = 100 \cdot m^2 \log(m) + 10 \cdot m^3$ grows asymptotically slower than $g(m) = 5 \cdot m^3 \log(m)$.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{100 \cdot m^2 \log(m) + 10 \cdot m^3}{5 \cdot m^3 \log(m)} \\ &= \lim_{m\to\infty} \frac{20}{m} + \frac{2}{\log(m)} = 0 + 0.\end{aligned}$$

(c) $f(m) = \log(m)$ grows asymptotically slower than $g(m) = \log(m^4)$.

Solution:

False, since

$$\begin{aligned}\lim_{m\to\infty}\frac{f(m)}{g(m)} &= \lim_{m\to\infty}\frac{\log(m)}{\log(m^4)}\\ &= \lim_{m\to\infty}\frac{\log(m)}{4\log(m)} = \frac{1}{4} > 0.\end{aligned}$$

(d) $f(m) = 2^{(0.9m^2+m)}$ grows asymptotically slower than $g(m) = 2^{(m^2)}$.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty}\frac{f(m)}{g(m)} &= \lim_{m\to\infty}\frac{2^{(0.9m^2+m)}}{2^{(m^2)}}\\ &= \lim_{m\to\infty} 2^{m-0.1m^2} = 0,\end{aligned}$$

as $m - 0.1m^2 \to -\infty$ as $m \to \infty$.

(e) If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any $a, b : \mathbb{N} \to \mathbb{R}^+$, if $\lim_{m\to\infty} a(m) = A$ and $\lim_{m\to\infty} b(m) = B$, then $\lim_{m\to\infty} a(m)b(m) = AB$.*

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty}\frac{f(m)}{h(m)} &= \lim_{m\to\infty}\frac{f(m)g(m)}{h(m)g(m)}\\ &= \lim_{m\to\infty}\frac{f(m)}{g(m)}\cdot\frac{g(m)}{h(m)}\\ &= \lim_{m\to\infty}\frac{f(m)}{g(m)}\cdot\lim_{m\to\infty}\frac{g(m)}{h(m)} = 0.\end{aligned}$$

(f) If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Solution:

False, consider $f(m) = 1/m^2$, $g(m) = 1/m$ and $h(m) = m^3$. They satisfy the conditions, but f does not grow slower than $g(h(m)) = 1/m^3$.

Exercise 1.5 *Proving Inequalities.*

(a) By induction, prove the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n} \le \frac{1}{\sqrt{3n+1}}, \quad n \ge 1.$$

Solution:

- **Base Case.**
  For $n = 1$:
  $$\frac{1}{2} \leq \frac{1}{\sqrt{4}},$$
  which is an equality.

- **Induction Hypothesis.**
  Now we assume that it is true for $n = k$, i.e.,
  $$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}.$$

- **Induction Step.**
  We will prove that it is also true for $n = k + 1$.
  $$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}$$
  .

  Plugging in the induction hypothesis, it is sufficient to prove.
  $$\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}} \quad \Leftrightarrow$$
  .
  $$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}$$
  .

  Rewriting:
  $$\begin{aligned}
  &\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k+1}{3k+4}} \\
  \Leftrightarrow\ &\left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4} \\
  \Leftrightarrow\ &(4k^2 + 4k + 1)(3k + 4) \leq (4k^2 + 8k + 4)(3k + 1) \\
  \Leftrightarrow\ &12k^3 + 28k^2 + 19k + 4 \leq 12k^3 + 28k^2 + 20k + 4 \\
  \Leftrightarrow\ &0 \leq k
  \end{aligned}$$

  Hence it is true for $n = k + 1$.

(b)* Replace $3n + 1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

Solution:

(b) Sometimes it is easier to prove more than less. This simple approach does not work for the weaker inequality as we are using a weaker (and insufficiently so!) induction hypothesis in each step.